JOURNAL OF ALGEBRA 39, 249–254 (1976)

Fitting Theory in a Class of Locally Finite Groups

A. A. KLIMOWICZ

Department of Pure Mathematics, University of Sheffield, Sheffield, S3 7RH England

Communicated by J. A. Green

Received October 4, 1974; revised December 16, 1974

1. INTRODUCTION

In [4], we introduced a class $\mathfrak{B}$ of periodic locally soluble groups and showed that $\mathfrak{B}$-groups possess many of the properties of finite soluble groups. In this paper, we investigate the class $\mathfrak{B}$ further but this time in relation to Fitting classes of $\mathfrak{B}$-groups. The main result in Section 2 is that any $\mathfrak{B}$-group possesses a unique local conjugacy class of $\mathfrak{F}$-injectors, for any Fitting class $\mathfrak{F}$ of $\mathfrak{B}$-groups. This extends Tomkinson's result [7] in periodic locally soluble FC-groups. In Section 3, we consider normal Fitting classes of $\mathfrak{B}$-groups and generalize the results of Blessenohl and Gaschütz [1] and Lausch [5] (in finite soluble groups). In particular, we show that every nontrivial normal Fitting class of $\mathfrak{B}$-groups admits one and only one normal Fitting pair (up to isomorphic Fitting pairs).

NOTATION AND TERMINOLOGY

If $\mathfrak{X}$ is a class of groups we denote by $\mathfrak{X}^*$ the class of finite $\mathfrak{X}$-groups. We denote by $\mathfrak{D}$, $\mathfrak{N}$ the classes of all groups, and periodic nilpotent groups, respectively.

Let $\mathfrak{X}$ be a class of groups, and let G be a group. Then, the *$\mathfrak{X}$-residual* $G^{\mathfrak{X}}$ of G is the intersection of all normal subgroups N of G with $G/N \in \mathfrak{X}$. Furthermore, if $\mathfrak{X}$ is R-closed, $G^{\mathfrak{X}}$ is the unique minimal normal subgroup of G such that $G/G^{\mathfrak{X}} \in \mathfrak{X}$. The *$\mathfrak{X}$-radical* $G_{\mathfrak{X}}$ of G is the group generated by all normal subgroups K of G with $K \in \mathfrak{X}$, and if $\mathfrak{X}$ is N-closed, then $G_{\mathfrak{X}}$ is the unique maximal normal $\mathfrak{X}$-subgroup of G.

We refer to [6] for the definitions of serial subgroups and the various local concepts used throughout, and to [3] for a discussion of inverse limits. The following result can be found in [3, 1.K.1]:

THEOREM 1.1. *The inverse limit of an inverse system of nonempty finite sets is nonempty.*

We shall be working throughout this paper in the class $\mathfrak{V}$ of groups G possessing a local system Σ of finite soluble subgroups such that $G^{\mathrm{L}\mathfrak{N}} \leq N_G(\Sigma) = \bigcap_{F\in\Sigma} N_G(F)$. Clearly, F is a serial subgroup of G, denoted by F ser G, for all $F \in \Sigma$, and we refer to [4] for the basic properties of $\mathfrak{V}$-groups. Throughout, when considering a V-group G, we shall assume that Σ denotes a local system with the above property.

2. FITTING CLASSES OF $\mathfrak{V}$-GROUPS

A *Fitting class* of $\mathfrak{V}$-groups is a subclass $\mathfrak{F}$ of $\mathfrak{V}$ such that

(i) if $G \in \mathfrak{F}$ and H ser G, then $H \in \mathfrak{F}$,

(ii) if $\{S_\lambda ; \lambda \in \Lambda\}$ is a family of serial $\mathfrak{F}$-subgroups of a $\mathfrak{V}$-group G such that $G = \langle S_\lambda ; \lambda \in \Lambda\rangle$, then $G \in \mathfrak{F}$.

The following is immediate from the definition of a Fitting class.

LEMMA 2.1. *If $\mathfrak{F}$ is a Fitting class of $\mathfrak{V}$-groups, then $\mathfrak{F}^*$ is a Fitting class of finite soluble groups.*

In this section, the proofs of several of the results are similar to the analogous results of [7] and, therefore, are omitted.

THEOREM 2.2. *If $\mathfrak{G}$ is a Fitting class of finite soluble groups, then $\mathrm{L}\mathfrak{G} \cap \mathfrak{V}$ is a Fitting class of $\mathfrak{V}$-groups.* (*cf.* [7, *Theorem* 2.2.])

It follows from the above results that the Fitting classes of $\mathfrak{V}$-groups are precisely the $\mathrm{L}\mathfrak{G} \cap \mathfrak{V}$, where $\mathfrak{G}$ is a Fitting class of finite soluble groups.

COROLLARY 2.3. *If $\mathfrak{F}$ is a Fitting class of $\mathfrak{V}$-groups, then $\mathrm{L}\mathfrak{F} \cap \mathfrak{V} = \mathfrak{F}$.*

If $\mathfrak{F}$ is a Fitting class of $\mathfrak{V}$-groups and $G \in \mathfrak{V}$, then an *$\mathfrak{F}$-injector* of G is a subgroup V of G such that $V \cap S$ is a maximal $\mathfrak{F}$-subgroup of S for all S ser G. Clearly, we have the following:

LEMMA 2.4. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{V}$-groups. If V is an $\mathfrak{F}$-injector of a $\mathfrak{V}$-group G, and H ser G, then $V \cap H$ is an $\mathfrak{F}$-injector of H.*

Our next result establishes the existence of $\mathfrak{F}$-injectors of a $\mathfrak{V}$-group for any Fitting class $\mathfrak{F}$ of $\mathfrak{V}$-groups. This generalizes [7, Theorem 3.2].

THEOREM 2.5. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{V}$-groups. Then, if G is a $\mathfrak{V}$-group, G possesses $\mathfrak{F}$-injectors.*

Proof. From Lemma 2.4, we see that the $\mathfrak{F}$-injectors of the subgroups of Σ form an inverse system of finite nonempty sets. By Theorem 1.1, the inverse limit is nonempty and the set-theoretic union V of one of its elements is easily seen to be an $\mathrm{L}\mathfrak{F}$-subgroup of G, and therefore, an $\mathfrak{F}$-subgroup by Corollary 2.3. That V is an $\mathfrak{F}$-injector of G follows without difficulty (see [7, Theorem 3.2]).

THEOREM 2.6 (cf. [7, Theorem 3.4]). *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{V}$-groups, and let V be an $\mathfrak{F}$-injector of a $\mathfrak{V}$-group G. If $V \leqslant H \leqslant G$, then V is an $\mathfrak{F}$-injector of H.*

LEMMA 2.7. *Let $\mathfrak{F}$ be a Fitting class of finite soluble groups, and suppose that G is a finite soluble group. Then, any two $\mathfrak{F}$-injectors of G are conjugate via the nilpotent residual of G.*

Proof. This is immediate if we note that the conjugating element can be chosen in $G^{\mathfrak{N}}$ in [2, Satz 1, p. 338].

We are now in a position to show that in a $\mathfrak{V}$-group the $\mathfrak{F}$-injectors form a unique local conjugacy class.

THEOREM 2.8. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{V}$-groups. Then, any two $\mathfrak{F}$-injectors of a $\mathfrak{V}$-group G are locally conjugate in G.*

Proof. Let V_1, V_2 be any two $\mathfrak{F}$-injectors of G, and let $\Sigma = \{F_\lambda ; \lambda \in \Lambda\}$. Define

$$\begin{aligned} A_\lambda = \{&\text{automorphisms of } F_\lambda \text{ induced by conjugation by elements of } G^{\mathrm{L}\mathfrak{N}}, \\ &\text{and which map } V_1 \cap F_\lambda \text{ onto } V_2 \cap F_\lambda\}. \end{aligned}$$

Then, A_λ is finite and nonempty for all $\lambda \in \Lambda$, by Lemma 2.7. Clearly, $\{A_\lambda ; \lambda \in \Lambda\}$ forms an inverse system of finite sets. By Theorem 1.1, we may choose an element $(\phi_\lambda) \in \operatorname{proj\,lim}\{A_\lambda\}$. Then, (ϕ_λ) defines a locally inner automorphism of G mapping V_1 onto V_2.

3. NORMAL FITTING CLASSES OF $\mathfrak{V}$-GROUPS

A Fitting class $\mathfrak{F}$ of $\mathfrak{V}$-groups is called a *normal* Fitting class of $\mathfrak{V}$-groups, if, for all $\mathfrak{V}$-groups G, the $\mathfrak{F}$-injectors of G are normal subgroups of G.

Clearly, if $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{V}$-groups, then $\mathfrak{F}^*$ is a normal Fitting class of finite soluble groups.

THEOREM 3.1. *If $\mathfrak{G}$ is a normal Fitting class of finite soluble groups, then* $\mathrm{L}\mathfrak{G} \cap \mathfrak{B}$ *is a normal Fitting class of $\mathfrak{B}$-groups.*

Proof. By Theorem 2.2, $\mathrm{L}\mathfrak{G} \cap \mathfrak{B}$ is a Fitting class of $\mathfrak{B}$-groups. Let $\mathfrak{F} = \mathrm{L}\mathfrak{G} \cap \mathfrak{B}$, G be a $\mathfrak{B}$-group, and let V be an $\mathfrak{F}$-injector of G. Let $v \in V$, $g \in G$, then, there exists $F \in \Sigma$ such that $v, g \in F$. By Lemma 2.5, $V \cap F$ is an $\mathfrak{F}$-injector of F, i.e., $V \cap F$ is a $\mathfrak{G}$-injector of F. Therefore, $v^g \in V \cap F$, i.e., V is a normal subgroup of G. Hence, $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{B}$-groups.

COROLLARY 3.2. *The normal Fitting classes of $\mathfrak{B}$-groups are precisely the* $\mathrm{L}\mathfrak{G} \cap \mathfrak{B}$, *where the $\mathfrak{G}$ run over the normal Fitting classes of finite soluble groups.*

From Cossey's result [1, Satz 5.1] and Corollary 2.3, we easily deduce:

THEOREM 3.3. *If $\mathfrak{F}$ is a nontrivial normal Fitting class of $\mathfrak{B}$-groups, then, the class of periodic locally nilpotent groups is contained in $\mathfrak{F}$.*

LEMMA 3.4. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{B}$-groups, and let K be a serial subgroup of a $\mathfrak{B}$-group G. Then, $K_{\mathfrak{F}} = K \cap G_{\mathfrak{F}}$.*

We are now able to extend [1, Satz 5.3].

THEOREM 3.5. *Let $\mathfrak{F}$ be a nontrivial Fitting class of $\mathfrak{B}$-groups. Then, $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{B}$-groups if and only if $G' \leqslant G_{\mathfrak{F}}$, for all $G \in \mathfrak{B}$.*

Proof. First, assume that $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{B}$-groups. Let G be a $\mathfrak{B}$-group and let $g, h \in G$. Then, there exists $F \in \Sigma$ such that $g, h \in F$, and so $[g, h] \in F' \leqslant G'$. Now, by Lemma 3.4, $F \cap G_{\mathfrak{F}} = F_{\mathfrak{F}}$. Therefore, by [1, Satz 5.3], $F' \leqslant F \cap G_{\mathfrak{F}}$. Hence, $G' \leqslant G_{\mathfrak{F}}$.

Conversely, suppose that G is a $\mathfrak{B}$-group. Then clearly, $G_{\mathfrak{F}} \leqslant V$ for all $\mathfrak{F}$-injectors V of G. Therefore, V is a normal subgroup of G, for all $\mathfrak{F}$-injectors V of G. Hence, $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{B}$-groups.

The following immediate corollary of Theorem 3.5 extends [1, Satz 6.2].

COROLLARY 3.6. *Let $\{\mathfrak{F}_i\,;\, i \in I\}$ be a family of nontrivial normal Fitting classes of $\mathfrak{B}$-groups. Then, $\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i$ is also a nontrivial normal Fitting class of $\mathfrak{B}$-groups.*

In the remainder of this section, we show that Lausch's theorem on normal Fitting pairs in finite soluble groups can be extended to the class of $\mathfrak{B}$-groups.

Let A be an arbitrary abelian group, let $\mathrm{Hom}(G, A)$ denote the set of group homomorphisms of G into A, where G is an arbitrary $\mathfrak{B}$-group, and let d associate with each $\mathfrak{B}$-group G an element of $\mathrm{Hom}(G, A)$.

The pair (A, d) is called a *normal Fitting pair* if

(N1) whenever G and H are $\mathfrak{V}$-groups, $\alpha: G \to H$ is a monomorphism and $G\alpha$ ser H, then $Gd = \alpha Hd$,

(N2) $A = \{(g)\, Gd;\, g \in G \in \mathfrak{V}\}$.

Let $\mathfrak{F}(A, d) = \{G \in \mathfrak{V};\ G = \ker(Gd)\}$.

By using an argument similar to that incorporated in the proof of [1, Satz 3.1], we have the following result.

LEMMA 3.7. *$\mathfrak{F}(A, d)$ is a normal Fitting class of $\mathfrak{V}$-groups, and if R is the $\mathfrak{F}(A, d)$-radical of a $\mathfrak{V}$-group G, then $R = \ker(Gd)$, for all $\mathfrak{V}$-groups.*

A normal Fitting class of $\mathfrak{V}$-groups $\mathfrak{F}$ is said to *admit a normal Fitting pair* (A, d), if $\mathfrak{F} = \mathfrak{F}(A, d)$.

Let (A, d) and (A_1, d_1) be normal Fitting pairs. We say that they are *isomorphic* if there exists an isomorphism $\beta: A \to A_1$ such that $d\beta = d_1$.

LEMMA 3.8. *Let (A, d) and (A_1, d_1) be normal Fitting pairs. Then, $\mathfrak{F}(A, d) = \mathfrak{F}(A_1, d_1)$ if and only if (A, d) and (A_1, d_1) are isomorphic.*

Proof. Define d^* and $d_1{}^*$ as follows, if G is a finite soluble group, then $Gd^* = Gd$ and $Gd_1{}^* = Gd_1$. Then clearly, (A, d^*) and $(A_1, d_1{}^*)$ are normal Fitting pairs of finite soluble groups, and

$$\mathfrak{F}(A, d^*) = \mathfrak{F}(A, d) \cap \mathfrak{D}^* = \mathfrak{F}(A_1, d_1) \cap \mathfrak{D}^* = \mathfrak{F}(A_1, d_1{}^*).$$

Therefore, by [5, Proposition 2.1], (A, d^*) and $(A_1, d_1{}^*)$ are isomorphic, i.e., there exists an isomorphism $\beta: A \to A_1$ such that $d^*\beta = d_1{}^*$.

Now, let G be a $\mathfrak{V}$-group defined by the local system Σ, say, and let $g \in G$. Then, there exists $F \in \Sigma$ such that $g \in F$. Therefore,

$$(g)\, Gd\beta = (g)\, Fd\beta = (g)\, Fd^*\beta = (g)\, Fd_1{}^* = (g)\, Fd_1 = (g)\, Gd_1,$$

since F ser G. Hence, $d\beta = d_1$, and the result follows.

We now give our generalization of Lausch's theorem.

THEOREM 3.9 (cf. [5, Theorem 2.4]). *Every nontrivial normal Fitting class of $\mathfrak{V}$-groups admits one and only one normal Fitting pair (up to isomorphic Fitting pairs).*

Proof. Let $\mathfrak{F}$ be a nontrivial normal Fitting class of $\mathfrak{V}$-groups, and let $\mathfrak{G} = \mathfrak{F}^*$. Then, $\mathfrak{G}$ is a nontrivial normal Fitting class of finite soluble groups. By [5, Theorem 2.4], there exists a normal Fitting pair (A, d^*) such that $\mathfrak{G} = \mathfrak{F}(A, d^*)$.

Define the action of d by the following: If G is a $\mathfrak{V}$-group defined by the local system Σ, and $g \in G$, then define $(g)\,Gd = (g)\,Fd^*$, where $g \in F \in \Sigma$. Clearly, this is well defined and $Gd \in \mathrm{Hom}(G, A)$, for all $\mathfrak{V}$-groups G.

(N1) Suppose that G, H are $\mathfrak{V}$-groups, α: $G \to H$ is a monomorphism and $G\alpha \operatorname{ser} H$. Let Σ and Σ^* be local systems that define G and H, respectively, and let $g \in G$. Then, there exists $E \in \Sigma$ such that $g \in E$. Therefore, $g\alpha \in E\alpha \leqslant F$, for some $F \in \Sigma^*$. Now, $G\alpha \operatorname{ser} H$, and so $E\alpha$ is subnormal in F. Therefore, $Ed^* = \alpha Fd^*$, by the finite case. Hence, $(g)\,Gd = (g)\,Ed^* = (g)\,\alpha Fd^* = (g)\,\alpha Hd$, for all $g \in G$. Therefore, $Gd = \alpha Hd$, as required.

(N2) Clearly, $A = \{(g)\,Gd;\, g \in G \in \mathfrak{V}\}$.

Therefore, (A, d) is a normal Fitting pair. We show that $\mathfrak{F} = \mathfrak{F}(A, d)$. Let $G \in \mathfrak{F}(A, d)$, then $G = \ker(Gd)$. Therefore, $G \in \mathrm{L}\mathfrak{G} \cap \mathfrak{V} \leqslant \mathrm{L}\mathfrak{F} \cap \mathfrak{V} = \mathfrak{F}$. Conversely, let G be an $\mathfrak{F}$-group, then, G is an $\mathrm{L}\mathfrak{G}$-group, and so $G = \ker(Gd)$, which implies that $G \in \mathfrak{F}(A, d)$. Hence, $\mathfrak{F} = \mathfrak{F}(A, d)$. The "only one" part follows from Lemma 3.8.

References

1. D. Blessenohl and W. Gaschütz, Über normale Schunk- und Fittingklassen, *Math. Z.* **118** (1970), 1–8.
2. B. Fischer, B. Hartley, and W. Gaschütz, Injectoren der endlichen auflösbaren Gruppen, *Math. Z.* **102** (1967), 337–339.
3. O. H. Kegel and B. A. F. Wehrfritz, "Locally Finite Groups," North-Holland, 1973.
4. A. A. Klimowicz, Sylow structure and basis normalizers in a class of locally finite groups, to appear.
5. H. Lausch, On normal Fitting classes, *Math. Z.* **130** (1973), 67–72.
6. D. J. S. Robinson, "Infinite soluble and nilpotent groups," *Queen Mary College Math. Notes, London*, 1968.
7. M. J. Tomkinson, $\mathfrak{F}$-injectors of locally soluble FC-groups, *Glasgow Math. J.* **10** (1969), 130–136.